Lecture #5

- Last time: cross product.
  - Allowed us to build perpendicular vectors to 2 given vectors.
  - Example:

$$\vec{u} = \langle 7, -1, 3 \rangle \quad , \quad \vec{v} = \langle -4, 9, 6 \rangle$$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 7 & -1 & 3 \\ -4 & 9 & 6 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 3 \\ 9 & 6 \end{vmatrix}(\vec{i}) - \begin{vmatrix} 7 & 3 \\ -4 & 6 \end{vmatrix}(\vec{j}) + \begin{vmatrix} 7 & -1 \\ -4 & 9 \end{vmatrix}(\vec{k})$$

$$= [(-1)(6) - (3)(9)](\vec{i}) - [(7)(6) - (-4)(3)](\vec{j})$$
$$+ [(7)(9) - (-4)(-1)](\vec{k})$$

$$= -33(\vec{i}) - 54(\vec{j}) + 59(\vec{k})$$

$$= \langle -33, -54, 59 \rangle$$

- Recall: proposition (Properties of the $\times$ product)
  - let $\vec{u}, \vec{v}, \vec{w} \in R^3$ and $c \in R$

Algebraic Properties

1. $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$
2. $(c\vec{u}) \times \vec{v} = c(\vec{u} \times \vec{v}) = \vec{u} \times (c\vec{v})$
3. $\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w})$
4. $(\vec{u} + \vec{v}) \times \vec{w} = (\vec{u} \times \vec{w}) + (\vec{v} \times \vec{w})$
5. $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$
6. $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$

- (Properties of $\times$ product cont.)

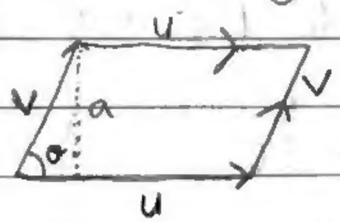
Geometric Properties

⑦ $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$

⑧ $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin(\theta)$, where θ is the angle between $\vec{u}$ and $\vec{v}$

⑨ $\vec{u} \times \vec{v} = \vec{0}$ iff $\vec{u}$ and $\vec{v}$ are parallel

- NB: The cross product obeys the right hand rule.
  - As for the magnitude,



Note that θ has $\sin(\theta) = \frac{a}{|\vec{v}|} \to a = |\vec{v}| \sin\theta$

$\therefore$ Area of parallelogram is: $A = (\text{altitude})(\text{base}) = a|\vec{u}| = |\vec{u}||\vec{v}| \sin\theta$

- Main point: the area of parallelogram formed by $\vec{u}$ and $\vec{v}$ is the magnitude of the cross product.

- Proof of part 8 of the proposition: we compute as follow

$|\vec{u} \times \vec{v}|^2 = (\vec{u} \times \vec{v}) \cdot \underbrace{(\vec{u} \times \vec{v})}_{\text{Pretend this: } \vec{w}\text{, then apply part 5.}}$ (property of dot product)

$= \vec{u} \cdot (\vec{v} \times (\vec{u} \times \vec{v}))$ apply property 6

$= \vec{u} \cdot ((\vec{v} \cdot \vec{v})\vec{u} - (\vec{v} \cdot \vec{u})\vec{v})$

- proof cont.

$$= \vec{u}\cdot((\vec{v}\cdot\vec{v})\vec{u}) - \vec{u}\cdot((\vec{v}\cdot\vec{u})\vec{v})$$

$$= (\vec{v}\cdot\vec{v})(\vec{u}\cdot\vec{u}) - (\vec{v}\cdot\vec{u})(\vec{u}\cdot\vec{v})$$

$$= |\vec{v}|^2 |\vec{u}|^2 - (\vec{u}\cdot\vec{v})^2$$

from properties of dot

$$= |\vec{u}|^2 |\vec{v}|^2 - (|\vec{u}||\vec{v}|\cos(\theta))^2$$

from geometric interpretation of dot product

$$= |\vec{u}|^2 |\vec{v}|^2 - |\vec{u}|^2 |\vec{v}|^2 \cos^2(\theta)$$

$$= |\vec{u}|^2 |v|^2 (1 - \cos^2(\theta))$$

$$= |\vec{u}|^2 |\vec{v}|^2 \sin^2(\theta)$$

$$= (|\vec{u}||\vec{v}|\sin(\theta))^2$$

$$\therefore (|\vec{u}| \times |\vec{v}|)^2 = (|\vec{u}||\vec{v}|\sin(\theta))^2$$

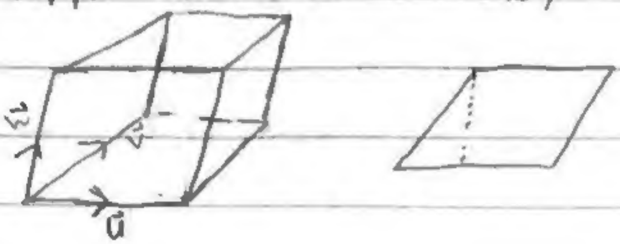
- On the other hand, θ is the geometric angle between $\vec{u}$ and $\vec{v}$

$\therefore$ θ can be expressed as $\theta \in [0, \pi]$

so $\sin(\theta) \geq 0$, Hence

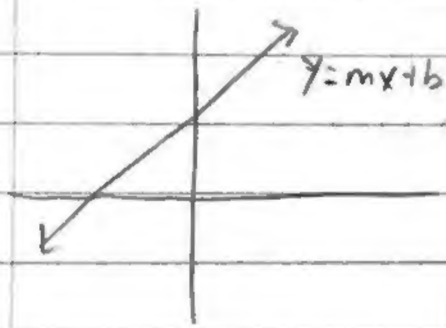
$|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}|\sin(\theta)$ as desired.

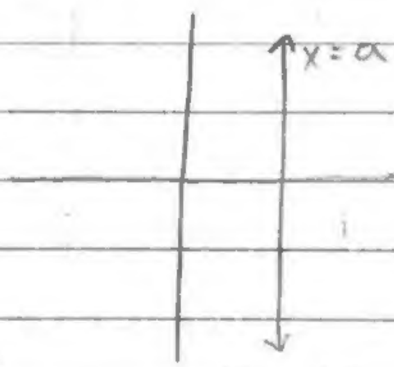
Cor: The <u>scalar triple product</u> $\vec{u} \cdot (\vec{v} \times \vec{w})$ computes the signed volume of the parallelepiped determined by $\vec{u}, \vec{v}, \vec{w}$



<u>§ 12.5: Lines and planes</u>

In 2-space





$ax + by = c$

$\vec{n} \cdot \langle x, y \rangle = c$

$(\vec{n} \neq \vec{0})$

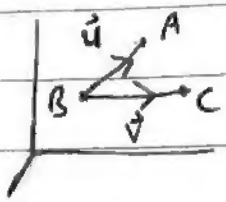
- In 3-space, examine $\vec{n} \cdot \vec{x} = d = \langle a, b, c \rangle \cdot \langle x, y, z \rangle$

$= ax + by + cz = d$

- This is a plane in 3-space

<u>NB</u>: Given 2 vectors, (both non-parallel), we get a plane. One normal vector to that plane is the cross product of the given vectors.

- Example: compute an equation of the plane containing the points:
$(0,1,3)$; $(2,4,0)$; $(1,2,3)$



Solution: note that the vectors

$\vec{u} = \langle 2-0,\ 4-1,\ 0-3 \rangle = \langle 2, 3, -3 \rangle$

$v = \langle 1-0,\ 2-1,\ 3-3 \rangle = \langle 1, 1, 0 \rangle$

• we can use a normal vector

$$\vec{n} = \vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & -3 \\ 1 & 1 & 0 \end{vmatrix} = (3, -3, -1)$$

$\therefore$ the plane has equation $\vec{n} \cdot \vec{x} = d$

• using $(0,1,3)$ and $3x - 3y - z = d$

we determine $d = 3 \cdot 0 - 3 \cdot 1 - 3 = -6$

$\therefore$ the plane has equation $3x - 3y - z = -6$